Lecture 11

lect 11

CHAIN RULE

Q. What is the derivative of the function $F(x) = \sqrt{x^2+1}$?

The formulas we have learned in previous section doesn't work.

Observe that F is a composite function.

let $y = f(u) = \sqrt{u}$

let $u = g(x) = x^2+1$

Then we can write $y = F(x) = f(g(x))$, that is $F = f \circ g$

We know how to differentiate both f and g so the question is how derivative of $F = f \circ g$ depends on the derivative of f and g.

This is where the Chain rule comes into play.

CHAIN RULE

If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex $F(x) = \sqrt{x^2 + 1}$. Find $F'(x)$.

Sol^n $F(x) = (f \circ g)(x) = f(g(x))$

where $f(u) = \sqrt{u}$, $g(x) = x^2 + 1$

Since, $f'(u) = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}}$ and $g'(x) = 2x$

Then $F'(x) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Alternate Sol^n

let $u = x^2 + 1$ and $y = \sqrt{u}$

$$F'(x) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

In using the chain rule, we work from outside to inside.

What we are doing is we differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function

$$\frac{d}{dx}\underbrace{f}_{\text{outer}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}}} = \underbrace{f'}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}}\underbrace{\cdot}_{\text{product}}\underbrace{g'(x)}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

<u>Ex</u> Differentiate a) $y = \sin(x^2)$

Outer function is the sine function.

Inner function is the squaring function.

Then by chain rule,

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin x}_{\substack{\text{outer}\\ \text{func}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{func}}} = \underbrace{\cos}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{func}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative of}\\ \text{inner func}}}$$

$$= 2x \cdot \cos(x^2)$$

b) Let $y = \sin^2 x = (\sin x)^2$

Outer function is the squaring function.

Inner function is the sine function.

So,

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)}_{\text{inner function}}^{2} = \underbrace{2}_{\text{derivative of outer function}} \cdot \underbrace{(\sin x)}_{\text{evaluated at inner function}} \cdot \underbrace{\cos x}_{\text{derivative of inner function}}$$

In Example a) we combined the Chain rule with the rule for differentiating the sine function. In general, if $y = \sin u$, where u is differentiable function of x, then by the Chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx}$$

Thus, $\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$

So we can differentiate trigonometric function by combining it with Chain Rule.

Special Case of Chain rule

Let $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$

By using the Chain rule and then the power rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot \frac{du}{dx} = n\,[g(x)]^{n-1} \cdot g'(x)$$

This is the power rule combined with Chain Rule

Ex Differentiate $y = (x^3 - 1)^{100}$

Taking $u = g[x] = x^3 - 1$ and $n = 100$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3-1)^{100} = 100\,(x^3-1)^{99}\,\frac{d}{dx}(x^3-1) = 100\,(x^3-1)^{99} \cdot 3x^2$$

$$= 300x^2 \cdot (x^3-1)^{99}$$

Ex Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$

STEP 1 Rewrite $f(x)$ as $f(x) = \dfrac{1}{(x^2+x+1)^{+1/3}} = (x^2+x+1)^{-1/3}$

Thus, $f'(x) = -\dfrac{1}{3}(x^2+x+1)^{-4/3} \cdot \dfrac{d}{dx}(x^2+x+1)$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

Ex Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

We are combining Power rule, Chain Rule first

Then, $g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$

Now we would like to use Quotient rule

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{(2t+1)\cdot(t-2)' - (t-2)(2t+1)'}{(2t+1)^2}$$

$$= \frac{9(t-2)^8}{(2t+1)^8} \cdot \frac{(2t+1)\cdot 1 - (t-2)\cdot 2}{(2t+1)^2}$$

$$= \frac{9(t-2)^8}{(2t+1)^8} \cdot \frac{2t+1-2t+4}{(2t+1)^2}$$

$$= \frac{9(t-2)^8}{(2t+1)^8} \cdot \frac{5}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Ex Differentiate $y = (2x+1)^5 . (x^3 - x + 1)^4$

Soln In this example we must use the product rule before using the Chain rule :

$$\frac{dy}{dx} = (2x+1)^5 \cdot \frac{d}{dx}\left[(x^3-x+1)^4\right] + (x^3-x+1)^4 \cdot \frac{d}{dx}\left[(2x+1)^5\right]$$

$$= (2x+1)^5 . 4(x^3-x+1)^3 \cdot \frac{d}{dx}(x^3-x+1) + (x^3-x+1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$

$$= 4(2x+1)^5 . (3x^2-1) . (x^3-x+1)^3 + 5(x^3-x+1)^4 . (2x+1)^4 . 2$$

$$= 2(2x+1)^4 . (x^3-x+1)^3 \left[2(2x+1).(3x^2-1) + 5(x^3-x+1)\right]$$

$$= 2(2x+1)^4 . (x^3-x+1)^3 \left[2(6x^3+3x^2-2x-1) + 5x^3-5x+5)\right]$$

$$= 2(2x+1)^4 . (x^3-x+1)^3 \left[17x^3+6x^2-9x+3\right]$$

It's called "Chain rule", when we make a longer chain by adding extra link.

Suppose $y = f(u)$, $u = g(t)$, $t = h(x)$

where f, g, h are diff. functions.

Then to compute the derivative of y with respect to x, we use the Chain rule twice

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}$$

Ex $f(x) = \underbrace{\sin}_{\text{outer}} \underbrace{(\cos(\tan x))}_{\text{inner}}$

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of}\\ \text{outside}}} \underbrace{(\cos(\tan x))}_{\substack{\text{evaluated at}\\ \text{inside}}} \cdot \frac{d}{dx}\left(\underbrace{\cos}_{\text{out}}(\underbrace{\tan x}_{\text{inside}})\right)$$

$$= \cos(\cos(\tan x)) \cdot \underbrace{(-\sin)}_{\substack{\text{derivative}\\ \text{of outside}}} \underbrace{(\tan(x))}_{\substack{\text{eval at}\\ \text{inside}}} \cdot \underbrace{\sec^2 x}_{\substack{\text{derivative}\\ \text{of inside}}}$$

$$= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \sec^2 x$$

Ex $y = \sqrt{\sec(x^3)} = (\sec(x^3))^{1/2}$

Then outside is square root, inside is $\sec(x^3)$

Then, $\frac{dy}{dx} = \frac{1}{2}(\sec(x^3))^{-1/2} \cdot \frac{d}{dx}(\underbrace{\sec}_{\text{outside}}(\underbrace{x^3}_{\text{inside}}))$

$$= \frac{1}{2\sqrt{\sec(x^3)}} \cdot \underbrace{\sec(x^3).\tan(x^3)}_{\substack{\text{derivative of} \\ \text{outside evaluated} \\ \text{at inside}}} \cdot \underbrace{3x^2}_{\substack{\text{derivative} \\ \text{of inside}}}$$

$$= \frac{3x^2 \cdot \sec(x^3).\tan(x^3)}{2\sqrt{\sec(x^3)}}$$

Ex $s = \frac{t}{\sqrt{1+t^2}} = \frac{t}{(1+t^2)^{1/2}}$

Then by quotient rule

$$\frac{ds}{dt} = \frac{(1+t^2)^{1/2} \cdot \frac{d}{dt}(t) - t \cdot \frac{d}{dt}(1+t^2)^{1/2}}{((1+t^2)^{1/2})^2}$$

$$\frac{ds}{dt} = \frac{(1+t^2)^{1/2} - t \cdot \frac{1}{\cancel{2}} \cdot (1+t^2)^{-\frac{1}{2}} \cdot \cancel{2}t}{(1+t^2)}$$

$$= \frac{\left(\sqrt{1+t^2} - \frac{t^2}{\sqrt{1+t^2}}\right)}{(1+t^2)} \cdot \frac{\sqrt{1+t^2}}{\sqrt{1+t^2}}$$

$$= \frac{(1+t^2) - t^2}{(1+t^2) \cdot (1+t^2)^{1/2}} = \frac{1}{(1+t^2)^{3/2}} = (1+t^2)^{-3/2}$$

<u>Ex</u> $y = \sin(\sin(\sin x)))$

$$\frac{dy}{dx} = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$$